MR4946075 05D05

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Intersection-union families. (English. English summary)

J. Combin. Theory Ser. A **217** (2026), *Paper No.* 106100, 23 *pp.*

A family $\mathcal{F}$ of subsets of $[n]$ is said to be *p-intersecting* if $|F \cap F'| \geq p$ for all $F, F' \in \mathcal{F}$, and it is said to be *q-union* if $|F \cup F'| \leq n - q$ for all $F, F' \in \mathcal{F}$. The authors consider families $\mathcal{F}$ that are both p-intersecting and q-union, called *(p,q)-families*.

G. O. H. Katona [in *Combinatorics (Proc. NATO Advanced Study Inst., Breukelen, 1974), Part 2: Graph theory; foundations, partitions and combinatorial geometry*, pp. 13–42, Math. Centre Tracts, No. 56, Math. Centrum, Amsterdam, 1974; MR0351890] considered the extremal problem of determining $m(n,p,q)$, the maximum size of a (p,q)-family over $[n]$. It is known that $m(n,1,1) = 2^{n-2}$ by the work of D. E. Daykin and L. Lovász [J. London Math. Soc. (2) **12** (1975/76), no. 2, 225–230; MR0384559] and P. D. Seymour [Mathematika **20** (1973), 208–209; MR0347619], and that $m(n,1,q) = m(n,q,1)$ was explicitly found by P. Frankl [J. Combinatorial Theory Ser. A **19** (1975), no. 2, 208–213; MR0376364] and D.-L. Wang [J. Combinatorial Theory Ser. A **23** (1977), no. 3, 344–348; MR0457230]. Frankl [*Extremal set systems*, Ph.D. thesis, Eötvös Loránd Univ., 1977] conjectured that the extremal (p,q)-families over $[n]$ are always attained as a product $\mathcal{G} \times \mathcal{H} := \{G \cup H : G \in \mathcal{G},\ H \in \mathcal{H}\}$, where $\mathcal{G} \subset 2^X$ is p-intersecting, $\mathcal{H} \subset 2^Y$ is q-union, and $X \cup Y$ is a partition of $[n]$. The conjecture is easily verified when $n-p-q=0$ or $n-p-q=1$, and it was proved by C. M. Bang, H. S. Sharp and P. M. Winkler [Discrete Math. **45** (1983), no. 1, 123–126; MR0700856] when $n-p-q=2$ or $n-p-q=3$.

In the paper under review, the authors find the exact value of $m(n,p,q)$ for sufficiently large n. Specifically, define the family

$$\mathcal{L}(n,a,b) := \{L \subset [n] : |L \cap [p+2a]| \geq p+a,\ |L \cap [n-q-2b+1,n]| \leq b\}.$$

One can check that $\mathcal{L}(n,a,b)$ is of the form $\mathcal{G} \times \mathcal{H}$ for a p-intersecting family $\mathcal{G}$ and q-union family $\mathcal{H}$. The authors prove the conjecture when $n-p-q=4$ by showing that if $\mathcal{F}$ is a (p,q)-family over $[n]$ where $n = p+q+4$, then

$$|\mathcal{F}| \leq \max\{|\mathcal{L}(n,2,0)|, |\mathcal{L}(n,1,1)|, |\mathcal{L}(n,0,2)|\}.$$

They also prove that, when $n-p-q=2r$, any (p,q)-family $\mathcal{F}$ over $[n]$ satisfies $|\mathcal{F}| \leq \max_{i+j=r} |\mathcal{L}(n,i,j)|$ for $n \geq 4r(r+1)^2$. Similarly, they show that when $n-p-q=2r+1$, any (p,q)-family $\mathcal{F}$ over $[n]$ satisfies $|\mathcal{F}| \leq \max_{i+j=r} |\mathcal{L}(n,i,j)|$ for $n \geq 8r(r+1)^2+6r$. In particular, this proves the conjecture for $n \geq (n-p-q+1)^3$.

The authors use the technique of shifting due to P. Erdős, C. Ko and R. Rado [Quart. J. Math. Oxford Ser. (2) **12** (1961), 313–320; MR0140419] to restrict the analysis to initial families without loss of generality. For the case when $n-p-q=2r$, they show that if a (p,q)-family $\mathcal{F}$ contains a *decisive set G of type* (a,b), i.e., a set G for which $|[p] \setminus G| = a$ and $|[n-q+1,n] \cap G| = b$, where $a+b=r$, then $\mathcal{F} \subset \mathcal{L}(n,a,b)$; otherwise, $|\mathcal{F}| < \max_{i+j=r} |\mathcal{L}(n,i,j)|$ for sufficiently large n. The case when $n-p-q=2r+1$ is a bit more subtle. If there are two decisive sets of different types in $\mathcal{F}$, then

$$\mathcal{F} \subset \mathcal{L}_0(n,a,b) := \{F \subset [n] : |F \cap [p+2a]| \geq p+a,\ |F \cap [n-q-2b-1,n]| \leq b+1\},$$

which is not an optimal (p,q)-family. If there are two decisive sets G_1, G_2 of the same type (a,b) with $p+2a+1 \in G_1$ and $p+2a+1 \notin G_2$, then $\mathcal{F} \subset \mathcal{L}(n,a,b)$. Otherwise, $|\mathcal{F}| < \max_{i+j=r} |\mathcal{L}(n,i,j)|$ for sufficiently large n. *Brahadeesh Sankarnarayanan*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.